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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. OCTOBER, 1898. No. 10.

ON THE GROUPS WHICH ARE DETERMINED BY A GIVEN GROUP.

By DR. G. A. MILLER.

A group (G) generally determines a number of different groups which throw much light on the structure of G. It is the object of this paper to study some of the properties and uses of these groups. When we transform each of the operators of G by any one of them the consequent permutation of these operators determines a substitution. If we use all the operators of any subgroup of G as transformers the corresponding substitutions will evidently form a group which has a 1, α isomorphism to the given subgroup, where the value of α generally depends upon the particular subgroup whose operators have been employed as transformers.

If we employ all the operators of G as transformers the corresponding substitutions form a group which has a 1, β isomorphism to G. This group has been called the *group of cogredient isomorphisms* of G. It is clear that β is the maximum value of α. Hence all the α's are equal to unity whenever $\beta=1$. In this case all the isomorphisms are said to be simple and this process does not lead to any group except those that are found in G.

When some of these isomorphisms are not simple the given process may lead to groups that are not contained in G. The group of lowest order in which this happens is the well known quaternion group of order 8. In this case the group of cogredient isomorphisms is the four-group, which is clearly not contained in the quaternion group. The groups which correspond to the transformations obtained by using the operators of a subgroup are clearly contained in the quaternion group. When the order of such a subgroup exceeds unity $\alpha=\beta=2$.

Since β operators of G must transform its operators according to the same substitution in its group of cogredient isomorphisms, G must contain just β operators that are commutative to each one of its operators. These constitute an Abelian characteristic* subgroup of G. This subgroup may be called the cogredient subgroup of G. Hence the factors of composition of G are the prime factors of β together with the factors of composition of its group of cogredient isomorphisms. The group of cogredient isomorphisms of the group of cogredient isomorphisms is called the second group of cogredient isomorphisms. From the preceding paragraph it follows that the second group of cogredient isomorphisms of the quaternion group is identity.

From the second group of cogredient isomorphisms we may obtain the third in the same manner as the second was obtained from the first, etc. If we arrive at identity by finding the successive groups of cogredient isomorphisms of a given group the group must be solvable since its factors of composition must be prime numbers. There are, however, many solvable groups which do not possess this property, *e. g.* the symmetric group of order 6. Hence the given condition is sufficient but not necessary for the solvability of a group.

G may have simple isomorphisms to itself which cannot be obtained by transforming it by its own operators. All such isomorphisms can be obtained by transforming G by operators that transform it into itself† and they correspond to a group known as the *group of isomorphisms* of G. This group contains the group of cogredient isomorphisms as a selfconjugate (not necessarily characteristic) subgroup. While the group of cogredient isomorphisms of G has a $1, \beta$ isomorphism to G the group of isomorphisms of G need not possess such a property; *e. g.* the group of isomorphisms of the four-group is the symmetric group of order 6.

The group of isomorphisms of G is very useful in the study of groups which contain G as a self-conjugate subgroup. Such a group must transform the operators of G according to its group of isomorphisms or according to some subgroup of this group. In substitution groups the group of isomorphisms is very useful to determine the number of intransitive groups that can be formed by making a group simply isomorphic to itself. Among the types of groups whose groups of isomorphisms have received considerable attention are the Abelian groups which contain no operator, besides identity, whose order differs from a given prime number,‡ the alternating and the symmetric group of any degree,‖ and the cyclical group of any order.§

We sometimes arrive at additional groups by finding the successive groups of isomorphisms of G, *i. e.* the group of isomorphisms of the group of isomorphisms, etc. In particular, it is well known that the group of isomorphisms of the group generated by three independent commutative operators of order 2 is the simple group of order 168 and that its second group of isomorphisms is the

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*Frobenius, *Berliner Sitzungsberichte*, 1895, page 183.

†Cf. Frobenius, *Berliner Sitzungsberichte*, 1895, page 184.

‡Cf. Moore, *Bulletin of the American Mathematical Society*, vol. 2, 1895, pages 33-43.

‖Cf. Hoelder, *Mathematische Annalen*, vol. 46, 1895, pages 333-345.

§Cf. Burnside's *Theory of Groups*, 1897, page 240.

group of order 336, which may be represented as a transitive group of degree 8. We do not obtain any additional group by finding the higher groups of isomorphisms since the group of isomorphisms of a simple group of composite order is necessarily complete.*

It may happen that a group is simply isomorphic to its group of cogredient isomorphisms as well as to its group of isomorphisms. Such a group has been called a complete group. The symmetric group of every degree except two and six is complete. The group of degree 4 and order 8 is an example of a group that is simply isomorphic to its group of isomorphisms without being simply isomorphic to its group of cogredient isomorphisms, while the alternating group of every degree except three is simply isomorphic to its group of cogredient isomorphisms without being simply isomorphic to its group of isomorphisms. The cyclical group of composite order is clearly not simply isomorphic to either of these groups of isomorphisms.

Every selfconjugate subgroup of a group may be regarded as a modulus with respect to which all of its operators may be divided into sets containing an equal number of operators. These sets of operators determine a group known as the quotient group of the given group with respect to the particular selfconjugate subgroup. We have seen that such a quotient group need not be simply isomorphic to any subgroup of the given group.

If s and t represent any two operators of the group (G), $s^{-1}t^{-1}st$ will also represent an operator of G. All the operators of G that can be represented in the form $s^{-1}t^{-1}st$ generate the commutator subgroup of G. This subgroup has also been called the first derivative of G. With respect to this derivative G is isomorphic to an Abelian group of maximum order. The necessary and sufficient condition that G is solvable is that we arrive at identity when we form its successive derivatives.† G may have several self-conjugate subgroups as well as several characteristic subgroups, but it can have only one commutator subgroup as well as only one cogredient subgroup. If a group coincides with its commutator subgroup it is said to be perfect.

If G is represented as a regular group the largest substitution group of the same elements that transforms G into itself determines the holomorph of G,‡ *i. e.* this substitution group is simply isomorphic to the holomorph of G. The subgroup which includes all the substitutions that do not contain any one element of this substitution group determines the group of isomorphisms of G. The holomorph of G may also be defined as the largest group that transforms G into itself and contains only as many operators as are contained in G that are commutative to every operator of G. When G is a complete group the order of its holomorph is the square of the order of G.

When G contains a subgroup that does not include any selfconjugate subgroup besides identity, it can be represented as a transitive substitution group whose degree is obtained by dividing the order of G by the order of this subgroup. When the subgroup is a maximum subgroup the corresponding transitive group is primitive; when this condition is not satisfied it is imprimitive*. All subgroups of this type that can be made to correspond in a simple isomorphism of G to itself, and only these, lead to the same transitive group. Hence we can readily obtain all the transitive substitution groups that are determined by G, *i. e.* those which are simply isomorphic to G.

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*Cf. Ibid, page 238.

†Miller, *American Journal of Mathematics*, vol. 20, 1898, page 277.

‡Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

From what precedes it follows that the complete study of a group (G) implies the study of its successive groups of isomorphisms and those of cogredient isomorphisms, its selfconjugate subgroups and the corresponding quotient groups, its successive derivatives and the corresponding solvable groups, its characteristic and cogredient subgroups, its subgroups that do not involve any selfconjugate subgroup besides identity and the corresponding transitive substitution groups, etc. From this standpoint each group generally determines a group complex whose various parts throw much light on the structure of the group.

Chicago, August, 1898.

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*Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

INFINITY, THE INFINITESIMAL, AND ZERO.

By HENRY HEATON, M. Sc., Atlantic, Iowa.

So much has been written upon this subject that I do not flatter myself that I can write anything new. I shall only attempt to point out a few of the things that have been written that to my mind plainly cannot be true, and with these errors in view endeavor to express the truth as I see it. If in so doing I fall into error I shall have the satisfaction of knowing that greater men have done the same.

In the *Analyst*, Vol. VIII., pages 105-113, Professor Judson has a very interesting article upon this subject in which he quotes freely from many distinguished authors many things that are plainly fallacious. Yet when he attempts to outline his view of the subject it seems to the writer that he blunders fully as badly as those whom he criticises.

He says: "(7) *A variable which decreases indefinitely and which by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinitesimal. Thus, read when $x=\circ$, when x decreases indefinitely or when x is an infinitesimal.

(8) If a be a constant the expression $a/\circ=\infty$ and $a/\infty=\circ$ are rigidly exact."